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Letter to the Editor

Static and dynamic responses of a rigid circular plate on a tensionless Winkler foundation

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1. Introduction

Plates on elastic foundations have been studied widely under the motivation of engineering design, especially in civil and mechanical engineering problems, such as mat and raft foundations and pavement slabs on soil. These types of problems are usually analyzed by assuming that the foundation reacts in compression as well as in tension. Although this assumption simplifies the problem considerably, it is questionable in many practical problems of civil and mechanical engineering. It is realistic to expect that the contact between the plate and the foundation is established only within the region where the plate penetrates into the foundation. Outside this region the plate remains above the foundation and does not interact with it. Consequently, the contact region is not known in advance and its extent depends on the geometry of the problem and on the configuration of the loading. In this case the problem becomes non-linear and it can be solved for the cases when the contact region has a simple geometry, such as a circle. However, for complicated contact region, the solution can be accomplished by using iterative methods. There are various studies dealing with beams and plates resting on a unilateral elastic foundation [1,2]. Solutions are given for circular plates on the tensionless foundation subjected to static loading mostly by applying approximate solution techniques to the non-linear governing equations of the problem [3–6]. The governing equations of the contact problems can also be derived by using a variational formulation. However, the numerical solution can be obtained by using approximate solution techniques [7]. On the other hand, when the problem is a dynamic one, i.e., oscillations of a plate on a unilateral foundation, the boundary between the contact and the lift-off regions of the plate depends on time. In this case the solution is accomplished by adopting step-wise solutions in the time domain by updating the boundary continuously [8]. Recently, the forced vibrations of a rigid circular plate supported by a tensionless Winkler support along the edge of the plate were studied by assuming that the plate is subjected to a uniformly distributed load and a concentrated

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load [9]. Presently, this recent study is extended by considering a rigid circular plate subjected to a uniformly distributed load and an off-centre concentrated load and resting on a tensionless Winkler foundation. Numerical results are presented in figures to illustrate the static behaviour as well as the dynamic oscillations of the plate, particularly focusing on the tensionless nature of the foundation. The problem is solved by assuming the Winkler foundation model which is used for many practical problems such as raft foundations, rails and underground pipes. However, the use of a two-parameter foundation model may lead to more accurate results provided that the model parameters can be determined precisely, which is quite a complicated task. As it will be seen, in the present problem the curve between the contact region and the lift-off region is a straight line. However, for a two parameter foundation it is a general curve which has to be determined iteratively. Since the present paper aims to focus on the dynamic behaviour of a plate on a tensionless foundation, the simplest foundation model is preferred to avoid unnecessary sophistications.

2. Statement of the problem

The system considered is given in Fig. 1. It is a circular rigid plate of radius a, of mass M subjected to a concentrated load $P(t)$ having an eccentricity B and a uniformly distributed load $Q(t)$. The plate is assumed to be resting on a tensionless Winkler foundation having a modulus K_f. Fig. 1 shows the plate partly penetrated into the foundation and partly lifted off the foundation. Since the loading and the geometry of the plate are symmetric with respect to the radial axis of the plate passing through the application point of the load P, the contact curve which separates the contact and the lift-off regions is a straight line perpendicular to the symmetry axis as shown in

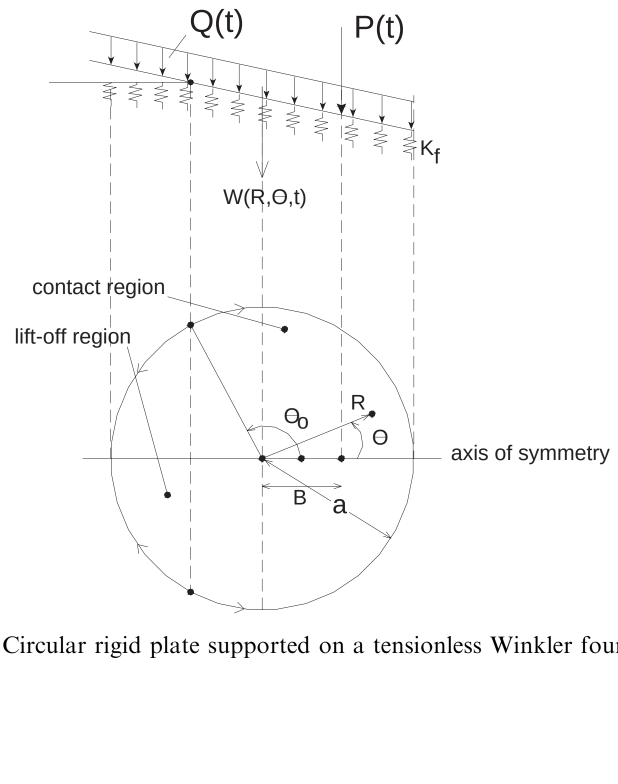

Fig. 1. Circular rigid plate supported on a tensionless Winkler foundation.

Fig. 1. The displacement function of the plate $W(R, \theta, t)$ will display the same symmetry and is composed of a rigid vertical translation and rotation along the horizontal axis perpendicular to the axis of symmetry. In fact, this simplifies the problem greatly. Thus, the equations of the translation and the rotation motions of the rigid plate can be expressed as

$$M \frac{\partial^2}{\partial t^2} W(R=0, \theta=0, t) = \pi a^2 Q + P - 2K_f \int_0^{\pi} \int_0^{a} H(R, \theta, t) W(R, \theta, t) R \, \mathrm{d}R \, \mathrm{d}\theta,$$

$$\tfrac{1}{4} M a^2 \frac{\partial^2}{\partial t^2} \left[\frac{\partial}{\partial R} W(R, \theta=0, t) \right]_{R=0} = PB - 2K_f \int_0^{\pi} \int_0^{a} H(R, \theta, t) W(R, \theta, t) R^2 \cos\theta \, \mathrm{d}R \, \mathrm{d}\theta. \quad (1)$$

By reason of symmetry, only a half of the plate is considered in the integrations. As usual the tensionless character of the elastic foundation under the plate is taken into consideration in Eq. (1), by introducing the contact function $H(R, \theta, t)$ defined as

$$H(R, \theta, t) = \begin{cases} 1 & \text{for } W(R, \theta, t) > 0, \\ 0 & \text{for } W(R, \theta, t) \leqslant 0. \end{cases} \quad (2)$$

The contact function takes care that the foundation exerts to the plate pressure only, when the plate penetrates into the foundation. The result of the foundation pressure and its moment is obtained by carrying the integration within the contact region only, which is ensured by the contact function. Due to the symmetry of problem, the translation and rotation of the rigid plate can be expressed as follows:

$$W(R, \theta, t) = a w(r, \theta, \tau) = a[R_0(\tau) + R_1(\tau) r \cos\theta], \quad (3)$$

where

$$r = R/a, \quad \tau = t\sqrt{g/a}$$

and $R_0(\tau)$ and $R_1(\tau)$ represent the non-dimensional rigid translation and rotation of the plate, respectively. The lift-off angle θ_0 is to be evaluated from

$$\theta_0(\tau) = \arccos\left[-\frac{R_0(\tau)}{R_1(\tau)} \right] \quad (4)$$

by assuming that $0 \leqslant \theta_0 \leqslant \pi$. The complete separation and the complete contact of the plate to the foundation correspond to the cases $\theta_0 = 0$ ($R_0/R_1 \leqslant -1$) and $\theta_0 = \pi$ ($R_0/R_1 \geqslant 1$), respectively. Substitution of the displacement function (3) into equations of motion (1) leads to the following system of two differential equations:

$$\mathbf{M}\ddot{\mathbf{R}} + \mathbf{K}\mathbf{R} = \mathbf{F}, \quad (5)$$

where the dots denote the differentiation with respect to the non-dimensional time τ and

$$\mathbf{R}^{\mathrm{T}} = [R_0(\tau), R_1(\tau)], \quad \mathbf{M} = [m_{ij}], \quad \mathbf{K} = [k_{ij}(\tau)], \quad \mathbf{F} = [f_i(\tau)],$$

$$m_{11} = 1, \quad m_{12} = m_{21} = 0, \quad m_{22} = 0.25, \quad (6)$$

$$k_{11}(\tau) = 2k_f \int_0^{\pi} \int_0^1 H(r, \theta, \tau) r \, \mathrm{d}r \, \mathrm{d}\theta,$$

$$k_{12}(\tau) = k_{21}(\tau) = 2k_f \int_0^{\pi} \int_0^1 H(r, \theta, \tau) \; r^2 \cos\theta \, \mathrm{d}r \, \mathrm{d}\theta,$$

$$k_{22}(\tau) = 2k_f \int_0^{\pi} \int_0^1 H(r, \theta, \tau) \; r^3 \cos^2\theta \, \mathrm{d}r \, \mathrm{d}\theta, \quad f_1(\tau) = q(\tau) + p(\tau), \quad f_2(\tau) = p(\tau)b.$$

The non-dimensional load and foundation parameters introduced are defined as

$$p = \frac{P}{Mg}, \quad q = \frac{\pi a^2 Q}{Mg}, \quad b = \frac{B}{a}, \quad k_f = \frac{K_f a^3}{Mg}. \tag{7}$$

Static and dynamic behaviour of the rigid plate is represented by the governing equation of problem (5), which represents the small amplitude motion of the plate. Due to the tensionless nature of the foundation, the stiffness matrix **K** is time-dependent and Eq. (5) is highly non-linear.

When a conventional Winkler foundation is assumed, it can be shown easily that the system has two equal free vibration periods such as

$$T_0 = T_1 = 2\sqrt{\frac{\pi}{k_f}}, \tag{8}$$

which correspond to both the vertical and rotational free vibrations of the rigid plate.

The static configuration of the rigid plate subjected to uniformly distributed load Q and the vertical off-centre load P can be studied easily by using the static version of Eq. (5)

$$\mathbf{KR} = \mathbf{F}. \tag{9}$$

Assuming that in the case of static equilibrium, the contact takes place for $0 \leqslant \theta \leqslant \theta_0$, as Fig. 1 shows, the elements of the matrix **K** can be evaluated as follows:

$$k_{11} = k_f[\theta_0 - 0.5 \sin 2\theta_0], \quad k_{12} = k_{21} = \tfrac{1}{6} k_f[9 \sin\theta_0 - \sin 3\theta_0],$$

$$k_{22} = 0.25 k_f[\theta_0 - 0.25 \sin 4\theta_0]. \tag{10}$$

An unexpected property of the static solution is reported in various similar studies [1–3]. The static solution is that the lift-off angle θ_0 does not depend on the magnitude of the loading but on the modulus of the foundation only. In the present case, the R_0 and R_1 depend linearly on the loading and the lift-off angle θ_0 does not depend on the level of the loading, when the plate is subjected to only one of the loading cases (q or p). On the other hand, when the two types of the loads are present—as it is in the present case—then the lift-off angle θ_0 will depend on the ratio of the loads $q/p = \pi a^2 Q/P$ and the foundation modulus. In these cases, where the lift-off angle θ_0 does not depend on the loading level itself but only on the ratio q/p, the global vertical equilibrium is established through the increase in the penetration of the rigid plate into the foundation and through the increase in the displacements, without any change in the extent of the contact region.

3. Numerical results and discussion

The effects of the parameters of the system on the behaviour of the plate are studied by obtaining various numerical results and presenting them in figures. The numerical procedure is verified for a number of special cases. It has been used to produce a limited set of results. When partial contact develops, the solution of the static case requires an iterative solution. On the other hand, the dynamic behaviour of the system is obtained by assuming an initial condition for the problem and by employing a step-wise numerical solution procedure for the governing differential equation (5) along the time axis. At each time step the contact angle θ_0 is updated according to the displacements of the plate at the previous time step and the elements of the matrix **K** are evaluated accordingly.

In the case of the static loading, the configuration of the plate is obtained assuming $k_f = 1.0$ and $p = 1.0$ for various values of the eccentricity b and that of the uniformly distributed load q. Figs. 2(a)–(c) show the variations of the lift-off angle θ_0, the non-dimensional vertical displacement R_0 and the rotation R_1, respectively. As it is seen, the lift-off comes into being, when the eccentricity of the vertical load increases and when the distributed load decreases. When no lift-off takes place, R_0 and R_1 are linear functions of the load q and the eccentricity b, respectively. However, the dependency becomes non-linear, when a lift-off appears and the vertical displacement and rotation increase rapidly. Figs. 3(a)–(c) display the similar variations for $k_f = 1.0$ and $q = 0.5$ for various values of the eccentricity b and the vertical load p. The inspection of the figures reveals that the full contact is established, when the eccentricity b or the load p decrease. The linear dependency of R_0 and R_1 on p and b can be observed, when no lift-off takes place. However, large and non-linear increases in R_0 and R_1 arise as a result of the lift-off of the plate from the foundation.

The present formulation does not have any restrictions concerning the time variation of the induced loads as well as the initial conditions of the system. However, for numerical evaluation, the initial configuration of the motion is assumed as the static equilibrium position of the plate under the load q. Oscillations of the plate starts through the instantaneous application of the vertical load. Under this assumption the dynamic behaviour of the system is presented in Fig. 4 for $b = 0.5$ and $k_f = 1.0$ assuming that the vertical load p is increased to 1.0 instantaneously. The time variation of the lift-off angle $\theta_0(\tau)$, the displacement $R_0(\tau)$ and the rotation $R_1(\tau)$ are displayed for various values of the load q. As Fig. 4(a) shows, the initial configuration of the plate is axially symmetric having full contact due to the symmetry of the initial loading. In the course of the oscillation, the full- and partial-contact cases follow each other for the present numerical combination of the parameters. The linear period of the system can be evaluated from Eq. (8) as $2\sqrt{\pi}$ for both vertical and rotational motions, assuming $k_f = 1$, provided that no lift-off takes place. However, the motion becomes highly non-linear, when the lift-off develops. Although the linear period of the motion can be identified in the oscillations of $R_0(\tau)$ and $R_1(\tau)$, these variations display a highly non-linear behaviour, when the lift-off takes place. Due to the partial contact the linear harmonic variations of the parameters vanish, the system softens and the observable period of the oscillations lengthens. Fig. 5 displays the variations $\theta_0(\tau)$, $R_0(\tau)$ and $R_1(\tau)$ for $q = 0.4$ and $k_f = 1.0$ for various values of the eccentricity b, assuming that the vertical load p is increased to 2.0 from zero instantaneously. Similar kind of time variations of these parameters of the problem can be seen in these figures. As the inspection of the figures reveals, the lift-off emerges more

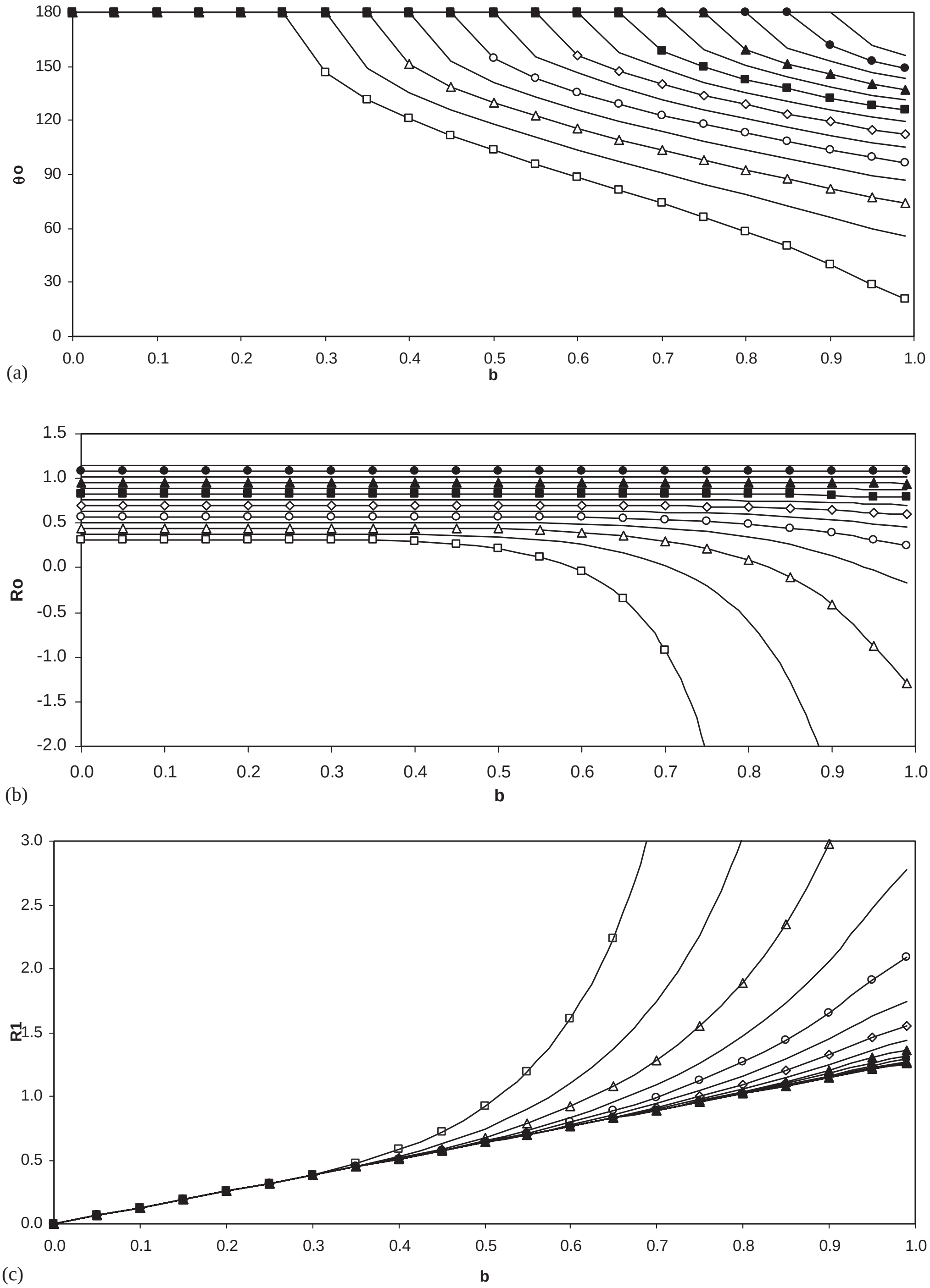

Fig. 2. Variations of: (a) θ_0 lift-off angle for $q/p = 0.0$, □; 0.4, △; 0.8, ◦; 1.2, ◇; 1.6, ■; 2.0, ▲; 2.4, ●; (b) R_0 vertical displacement; and (c) R_1 rotation depending on b eccentricity for $q = 0.0$, □; 0.4, △; 0.8, ◦; 1.2, ◇; 1.6, ■; 2.0, ▲; 2.4, ●; and for $k_f = 1.0$, $p = 1.0$.

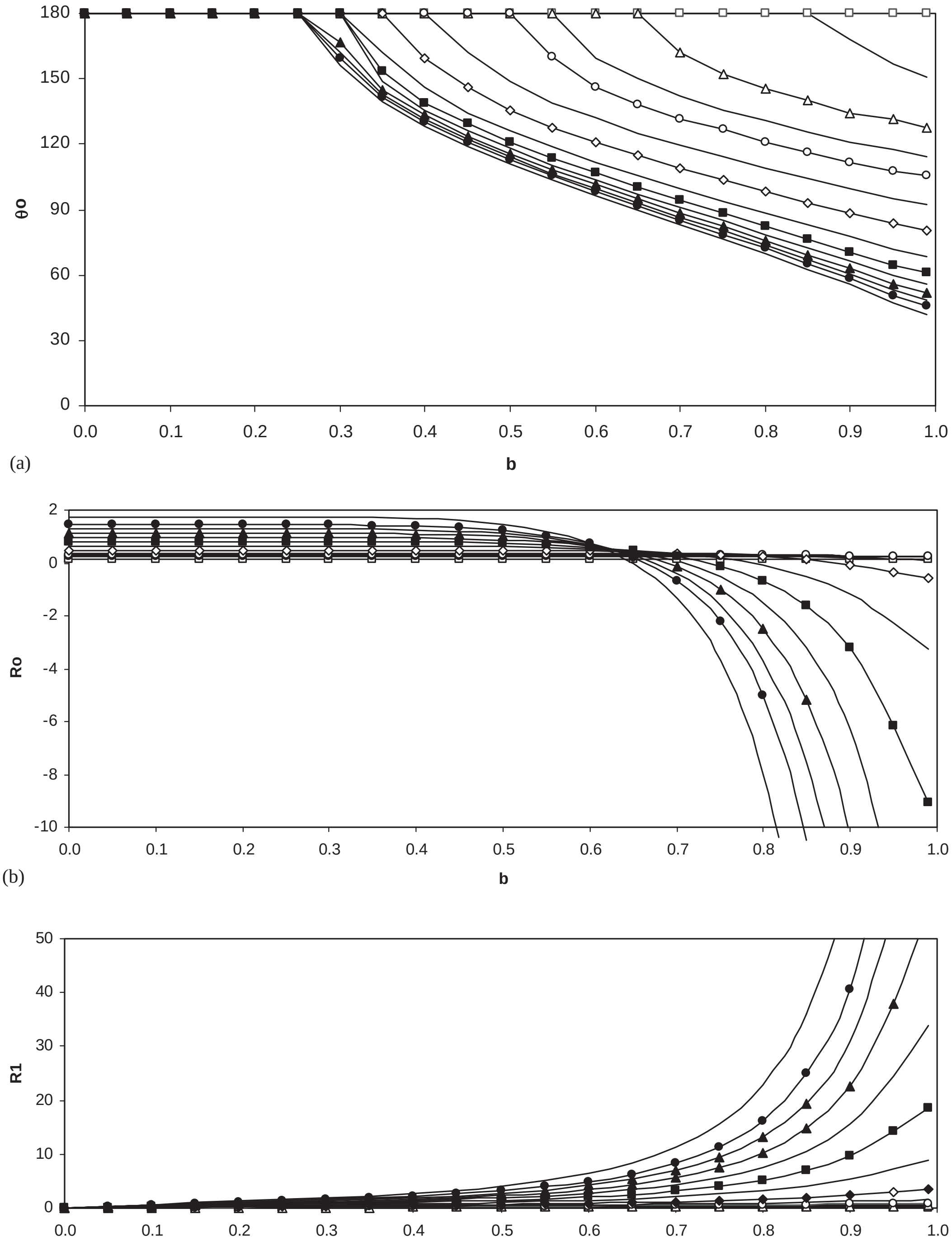

Fig. 3. Variations of: (a) θ_0 lift-off angle for $p/q = 0.0$, □; 0.3, △; 0.5, ◦; 1.0, ◇; 2.0, ■; 3.0, ▲; 4.0, ●; (b) R_0 vertical displacement; and (c) R_1 rotation depending on b eccentricity for $p = 0.0$, □; 0.3, △; 0.5, ◦; 1.0, ◇; 2.0, ■; 3.0, ▲; 4.0, ●; and for $k_f = 1.0$, $q = 0.5$.

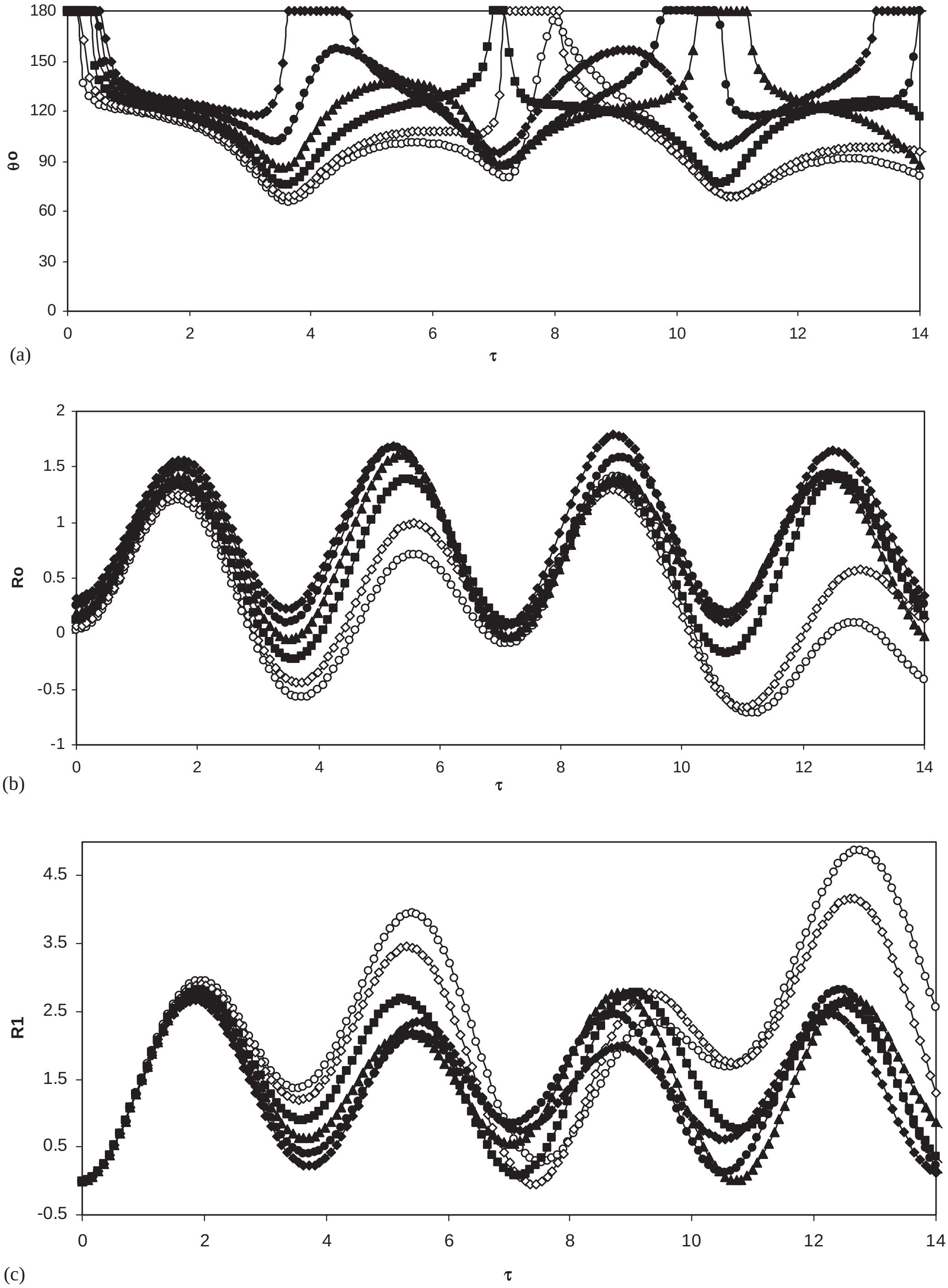

Fig. 4. Time variations of (a) $\theta_0(\tau)$ lift-off angle; (b) $R_0(\tau)$ vertical displacement; and (c) $R_1(\tau)$ rotation for $k_f = 1.0$, $b = 0.5$, $p = 0.0$ (1.0); and for $q = 0.1$, ◦; 0.20, ◇; 0.4, ■; 0.6, ▲; 0.8, ●, 1.0, ◆.

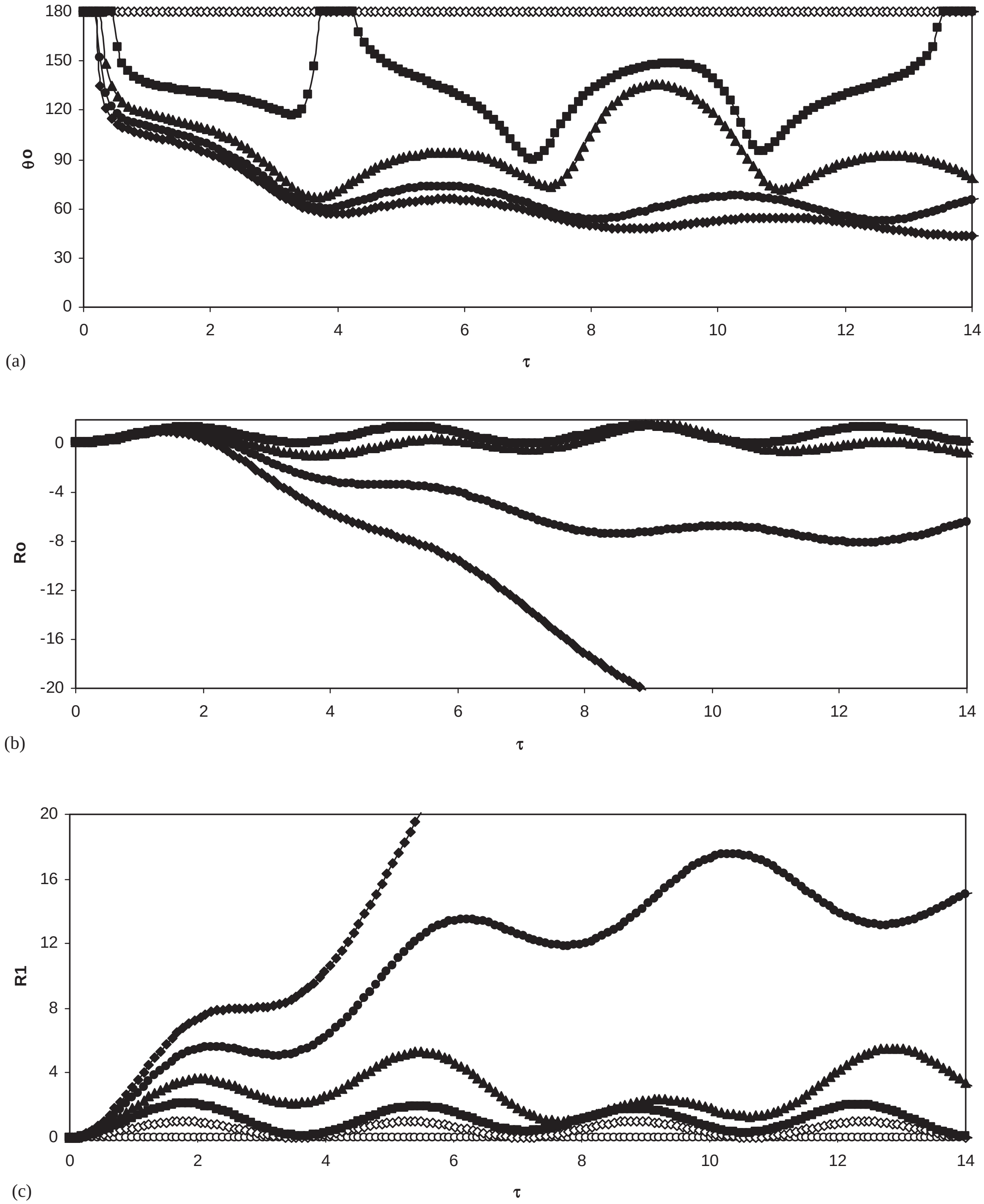

Fig. 5. Time variations of (a) $\theta_0(\tau)$ lift-off angle; (b) $R_0(\tau)$ vertical displacement and (c) $R_1(\tau)$ rotation for $k_f = 1.0$, $q = 0.0$, $p = 0.0$ (2.0); and for $b = 0.0$, ◦; 0.2, ◇; 0.4, ■; 0.6, ▲; 0.8, ●, 1.0, ◆.

distinct and the non-linearity appears much more pronounced, as the eccentricity of the load p is increased.

4. Conclusions

The static behaviour and forced oscillations of a rigid circular plate supported by a tensionless Winkler elastic foundation have been studied by assuming that the plate is subjected to a uniformly distributed load and a vertical load having an eccentricity. When a partial contact takes place due to the tensionless nature of the foundation, the governing equations and consequently the behaviour of the plate are non-linear, although the displacements of the plate are assumed to be small. The static configuration of the plate is evaluated as a special case. Numerical results are given in figures to determine the effects of the system parameters on the dynamic behaviour of the plate. It is seen that lift-off has a significant effect on the motion of the plate, the recognizable period of the oscillations is lengthened and the amplitudes become larger, because the tensionless foundation model is relatively less constrained compared to the conventional one.

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